Relational algebraic ornaments

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Internalism Externalism

Internalism

```
_++_ : Vec A m → Vec A n → Vec A (m + n)
[]        ++ ys = ys
(x :: xs) ++ ys = x :: (xs ++ ys)
```

proof structure follows program structure

How far can internalism go?

Minimum Coin Change

2/6 2/- 1/- 6d 3d 1d

shillings pence (d)

30d 24d 12d 6d 3d 1d

48d

30d 12d 6d 24d 24d

50¢ 25¢ 10¢ 5¢ 1¢

48¢

25¢ 10¢ 10¢ 1¢ 1¢ 1¢

specification

global optimisation

proved to meet

proof obligations about

“precisely typed” program

local optimisation

specification

global optimisation

? derive / reduce to

precise type

local optimisation

✔ develop interactively

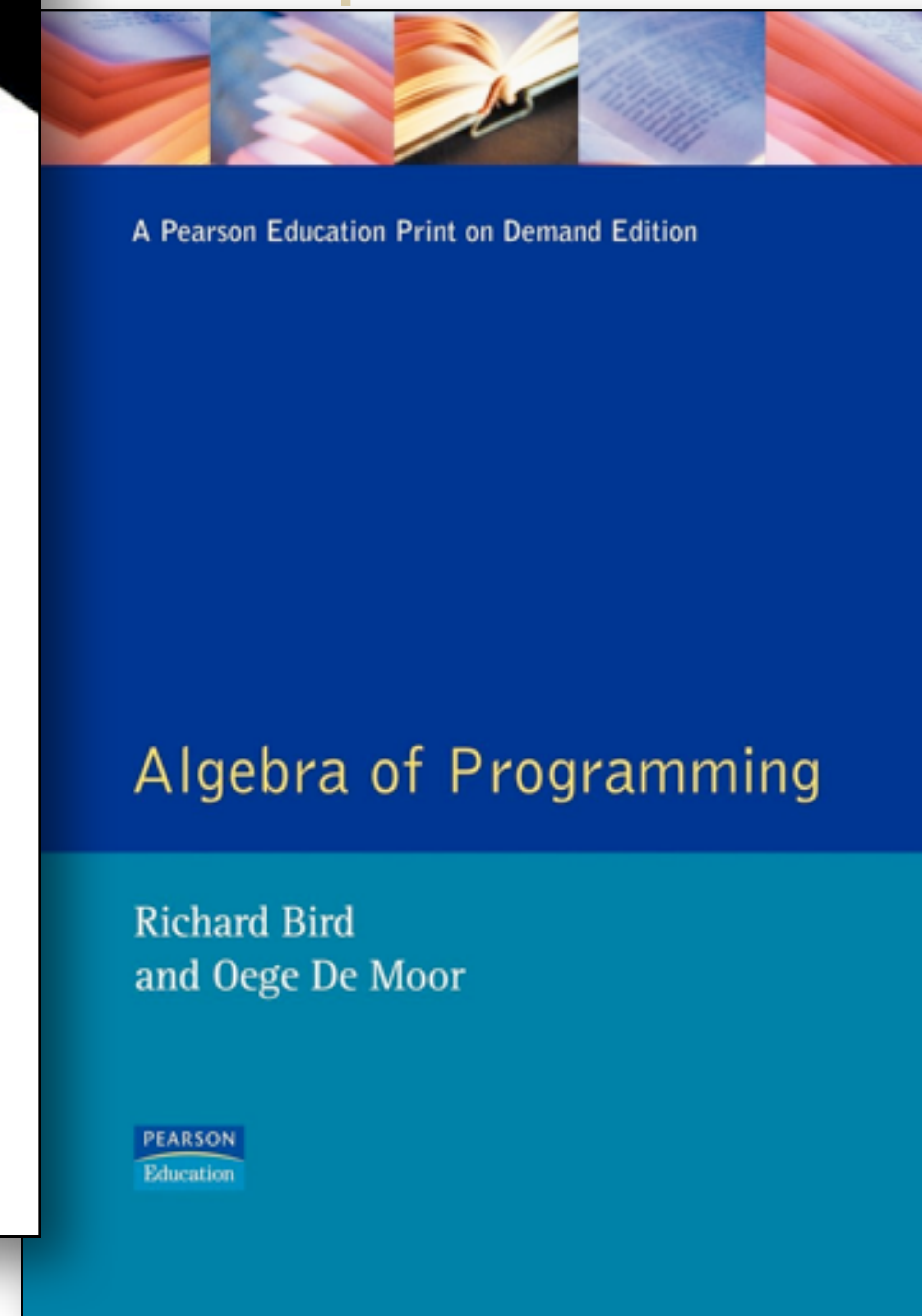
precisely typed program

need only exist; no inspection of the program tion

lobal optimisation

?

✓



relational specification

global optimisation

↓ refine by relational calculation

relational fold °

local optimisation

↓ derive by algebraic ornamentation

precise type

local optimisation

✔ develop interactively ↓

precisely typed program

Relations

potentially partial and nondeterministic mappings
(generalising functions)

$$A \to B \to \mathsf{Set}$$

predicates on (subsets of) $A \times B$

$R : A \to B \to \mathsf{Set}$ relates a to b
if $R\ a\ b : \mathsf{Set}$ is inhabited

Relations

potentially partial and nondeterministic mappings
(generalising functions)

```
A → (B → Set)
```

functions from A to subsets of B

Relations

potentially partial and nondeterministic mappings
(generalising functions)

$$A \rightsquigarrow B$$

relational programs from A to B

$\mathsf{R} : \mathsf{A} \rightsquigarrow \mathsf{B}$ nondeterministically maps a to b
if $\mathsf{R}\ a\ b : \mathsf{Set}$ is inhabited

inclusion

specification

$$
\begin{array}{ll}
& \mathit{ordered} \cdot \mathit{perm} \\
\supseteq & \quad \{\text{since } \mathit{flatten} \text{ is a function}\} \\
& \mathit{ordered} \cdot \mathit{flatten} \cdot \mathit{flatten}^\circ \cdot \mathit{perm} \\
= & \quad \{\text{claim: } \mathit{ordered} \cdot \mathit{flatten} = \mathit{flatten} \cdot \mathit{inordered} \text{ (see below)}\} \\
& \mathit{flatten} \cdot \mathit{inordered} \cdot \mathit{flatten}^\circ \cdot \mathit{perm} \\
= & \quad \{\text{converses}\} \\
& \mathit{flatten} \cdot (\mathit{perm} \cdot \mathit{flatten} \cdot \mathit{inordered})^\circ \\
\supseteq & \quad \{\text{fusion, for an appropriate definition of } \mathit{split}\} \\
& \mathit{flatten} \cdot (\!|\mathit{nil}, \mathit{split}^\circ|\!)^\circ .
\end{array}
$$

towards an executable program

Converse

$R : A \rightsquigarrow B = A \to B \to Set$

$R^{\circ} = flip\ R : B \rightsquigarrow A$

running R backwards

Relational folds

functional	relational
$f : 1 + A \times B \to B$	$S : 1 + A \times B \rightsquigarrow B$
$\mathrm{fold}\ f : \mathrm{List}\ A \to B$	$(\!\| S \|\!) : \mathrm{List}\ A \rightsquigarrow B$

$B = \mathrm{List}\ A$

$$
\begin{aligned}
S\ (\mathrm{inl}\ \_) &= \{\ [\,]\ \} \\
S\ (\mathrm{inr}\ (x\ ,\ xs)) &= \{\ xs\ ,\ x :: xs\ \}
\end{aligned}
$$

$\Rightarrow$ $(\!\| S \|\!)$ computes a subsequence of its input

Converse of relational folds

well-founded unfolds (generating inductive data)

```
sum  : List Nat ⇝ Nat

sum° : Nat ⇝ List Nat
```

breaks n into a (finite) list summing to n

Minimisation

generate all possible results of T

$$\mathtt{min}\ R \cdot \Lambda\ T$$

choose a minimum under R

T = the relation that nondeterministically breaks n into a list of coins representing n

R = the length ordering on lists

Greedy Theorem

$$\mathtt{min}\ R \cdot \Lambda\ (\!|\, S \,|\!)^{\circ} \quad S'$$
$$\supseteq\ (\!|\, \boxed{(\mathtt{min}\ Q \cdot \Lambda\ S^{\circ})^{\circ}} \,|\!)^{\circ}$$

if there exists Q such that ...

the minimum coin change problem can be solved by repeatedly choosing the largest possible denomination

$$\mathit{min}\ R \cdot \Lambda\, (\!| S |\!)^{\circ} \supseteq (\!| S' |\!)^{\circ}$$

$$p : \mathit{Nat} \to \mathit{List}\ \mathit{Coin}$$

same structure

$$(\!| S' |\!)^{\circ}\ n\ (p\ n)$$

$\Rightarrow$ { Greedy Theorem }

$$(\mathit{min}\ R \cdot \Lambda\, (\!| S |\!)^{\circ})\ n\ (p\ n)$$

Algebraic ornamentation

$S : 1 + A \times B \rightsquigarrow B$

$\mathsf{data}\ \mathrm{AlgList}\ S : B \to \mathrm{Set}$

$\mathrm{AlgList}\ S\ b \cong (xs : \mathrm{List}\ A) \times (\!| S |\!)\ xs\ b$

```
S : 1 + A × B  ⇝  B
AlgList S b  ≅  (xs : List A) × ⦇S⦈ xs b

data AlgList S : B → Set where
 nil   : {b : B} → S (inl tt) b →
         AlgList S b
 cons : {b : B} →
         (x : A) →
         {b' : B} → S (inr (x , b')) b →
         AlgList S b' → AlgList S b
```

```
AlgList S′ : Nat → Set
```

indexed by total value

the head of a nonempty list can only be the largest possible denomination

```
greedy : (n : Nat) → AlgList S′ n
```

$$\mathtt{greedy} : (\mathtt{n} : \mathtt{Nat}) \to \mathtt{AlgList}\ \mathtt{S'}\ \mathtt{n}$$

$$\boxed{\mathtt{AlgList}\ \mathtt{S'}\ \mathtt{n}} \cong (\mathtt{xs} : \boxed{\mathtt{List}\ \mathtt{Coin}}) \times (\!|\ \mathtt{S'}\ |\!)\ \mathtt{xs}\ \mathtt{n}$$

forget

$$(\!|\ \mathtt{S'}\ |\!)\ (\mathtt{forget}\ (\mathtt{greedy}\ \mathtt{n}))\ \mathtt{n}$$

$$\mathtt{p} = \mathtt{forget} \circ \mathtt{greedy} : \mathtt{Nat} \to \mathtt{List}\ \mathtt{Coin}$$

$$\mathsf{greedy} : (n : \mathsf{Nat}) \to \mathsf{AlgList}\ S'\ n$$

$$p = \mathsf{forget} \circ \mathsf{greedy} : \mathsf{Nat} \to \mathsf{List}\ \mathsf{Coin}$$

$$(\!|\, S' \,|\!)\ (p\ n)\ n$$

$\Rightarrow$ { converse }

$$(\!|\, S' \,|\!)^{\circ}\ n\ (p\ n)$$

$\Rightarrow$ { Greedy Theorem }

$$(\mathsf{min}\ R \cdot \Lambda\ (\!|\, S \,|\!)^{\circ})\ n\ (p\ n)$$

relational specification $\quad \mathrm{min}\ R \cdot \Lambda\ (\!|\ S\ |\!)^{\circ}$

$\downarrow$ relational program derivation

relational fold $^{\circ}$ $\quad (\!|\ S'\ |\!)^{\circ}$

$\downarrow$ algebraic ornamentation

internalist type $\quad$ `AlgList S′`

$\downarrow$ interactive development

internalist program $\quad$ `(n : Nat) → AlgList S′ n`

[Internalist type] derivation

Relational program derivation being one possible way